Endogenous timing and manufacturer advertising: A note

Mathematica Appendix

Qing Hu* and Tomomichi Mizuno+

*Faculty of Economics, Kushiro Public University of Economics, Ashino 4-1-1, Kushiro-shi, Hokkaido, 085-8585, Japan. E-mail: huqing549@gmail.com

+Graduate School of Economics, Kobe University, 2-1 Rokkodai, Nada, Kobe, Hyogo, 657-8501, Japan. E-mail: mizuno@econ.kobe-u.ac.jp


```
Clear["Global`*"]
```

クリア

2. Model

We define each consumer's demand as follows.

$$\text{demand} = \left\{q1 \to -\frac{a - p1 - a\gamma + p2\gamma}{b(-1+\gamma)},\ q2 \to -\frac{a - p2 - a\gamma + p1\gamma}{b(-1+\gamma)}\right\}$$

$$\left\{q1 \to -\frac{a - p1 - a\gamma + p2\gamma}{b(-1+\gamma)},\ q2 \to -\frac{a - p2 - a\gamma + p1\gamma}{b(-1+\gamma)}\right\}$$

Since θ consumers watching the advertising, demands for the retailers are

```
{θ q1, θ q2} /. demand;
Demand = {Q1 → %[[1]], Q2 → %[[2]]}
```

$$\left\{Q1 \to -\frac{(a - p1 - a\gamma + p2\gamma)\theta}{b(-1+\gamma)},\ Q2 \to -\frac{(a - p2 - a\gamma + p1\gamma)\theta}{b(-1+\gamma)}\right\}$$

Consumer surplus is

$$CS = \frac{\left(-p1^2 - p2^2 + 2a^2(-1+\gamma) - 2a(p1+p2)(-1+\gamma) + 2\,p1\,p2\,\gamma\right)\theta}{2b(-1+\gamma)}$$

$$\frac{\left(-p1^2 - p2^2 + 2a^2(-1+\gamma) - 2a(p1+p2)(-1+\gamma) + 2\,p1\,p2\,\gamma\right)\theta}{2b(-1+\gamma)}$$

The profits of retailers are

```
π1 = (p1 - w) Q1;
π2 = (p2 - w) Q2;
```

The profit of manufacturer is

```
πM = (w - c) (Q1 + Q2) - (k θ^2);
```

The producer and total surpluses are

```
PS = π1 + π2 + πM;
TS = CS + PS;
```

3. Analysis

Stage 4: simultaneous pricing

First, we consider the case with simultaneous pricing.
From the first-order condition, we obtain p^B in (1).

```
{π1, π2};
% /. Demand;
Solve[
解く
    {D[%[[1]], p1] == 0, D[%[[2]], p2] == 0}, {p1, p2}
     微分係数              微分係数
  ] // Flatten;
      平滑化
OutcomepB = %
```

$$\left\{p1 \to -\frac{a + w - a\gamma}{-2+\gamma},\ p2 \to -\frac{a + w - a\gamma}{-2+\gamma}\right\}$$

Stage 4: sequential pricing

Next, we consider the case with sequential pricing.
From the first-order condition, we obtain the best response for the $p^F(p^L)$ in (2):

```
π2;
% /. Demand;
Solve[D[%, p2] == 0, p2] // Flatten;
解く  微分係数                平滑化
OutcomepF = %;

% /. {p1 → pL}
```

$$\left\{p2 \to \frac{1}{2}\,(a + w - a\gamma + pL\,\gamma)\right\}$$

Then, the leader chooses the price p^L in (3):

```
π1;
% /. Demand;
% /. OutcomepF;
Solve[D[%, p1] == 0, p1] // Flatten // Simplify;
解く 微分係数 平滑化 簡単な形式に
OutcomepL = %
```

$$\left\{p1 \to \frac{w\left(-2-\gamma+\gamma^2\right)+a\left(-2+\gamma+\gamma^2\right)}{2\left(-2+\gamma^2\right)}\right\}$$

Stage 3

Proof of Lemma 1

Under simultaneous pricing in the retail market, the manufacturer chooses the following wholesale price.

```
πM;
% /. Demand;
% /. OutcomepB;
Solve[D[%, w] == 0, w] // Flatten // Simplify;
解く 微分係数 平滑化 簡単な形式に
Outcomew = %
```

$$\left\{w \to \frac{a+c}{2}\right\}$$

Under sequential pricing, the manufacturer sets the following wholesale price.

```
πM;
% /. Demand;
% /. OutcomepF;
% /. OutcomepL;
Solve[D[%, w] == 0, w] // Flatten // Simplify
解く 微分係数 平滑化 簡単な形式(
```

$$\left\{w \to \frac{a+c}{2}\right\}$$

Hence, we obtain Lemma 1. **Q.E.D.**

Stage 2

Proof of Lemma 2

Under simultaneous pricing in the retail market, the first-order condition leads to the advertising level θ^B in (4):

```
πM;
% /. Demand;
% /. OutcomepB;
% /. Outcomew;
Solve[D[%, θ] == 0, θ] // Flatten // Factor;
解く  微分係数           平滑化      因数分解
OutcomeθB = %

θ /. %;
θB = %
```

$$\left\{\theta \to -\frac{(a-c)^2}{4\,b\,k\,(-2+\gamma)}\right\}$$

$$-\frac{(a-c)^2}{4\,b\,k\,(-2+\gamma)}$$

Similarly, for the sequential pricing case, the manufacturer chooses the advertising level θ^S in (5):

```
πM;
% /. Demand;
% /. OutcomepF;
% /. OutcomepL;
% /. Outcomew;
Solve[D[%, θ] == 0, θ] // Flatten // FullSimplify;
解く  微分係数           平滑化      完全に簡約
OutcomeθS = %

θ /. %;
θS = %
```

$$\left\{\theta \to \frac{(a-c)^2\,(-8+(-1+\gamma)\,\gamma\,(4+\gamma))}{32\,b\,k\,(-2+\gamma^2)}\right\}$$

$$\frac{(a-c)^2\,(-8+(-1+\gamma)\,\gamma\,(4+\gamma))}{32\,b\,k\,(-2+\gamma^2)}$$

Here, we compare θ^B with θ^S.

```
θB - θS // Factor
          因数分解
```

$$-\frac{(a-c)^2\,(-1+\gamma)\,\gamma^2\,(2+\gamma)}{32\,b\,k\,(-2+\gamma)\,(-2+\gamma^2)}$$

Therefore, we obtain $\theta^B > \theta^S$. **Q.E.D.**

Here, we present the retailers’ profits in the case with simultaneous and sequential pricing as in (6) - (8).

```
π1;
% /. Demand;
% /. OutcomepB;
% /. Outcomew;
% /. OutcomeθB // Simplify;
                    簡単な形式に
```

```
πB = %
```

$$\frac{(a-c)^4 (-1+\gamma)}{16\, b^2\, k\, (-2+\gamma)^3}$$

```
{π1, π2};
% /. Demand;
% /. OutcomepF;
% /. OutcomepL;
% /. Outcomew;
% /. OutcomeθS // FullSimplify;
                    完全に簡約
```

```
{πL, πF} = %
```

$$\left\{\frac{(a-c)^4 (-1+\gamma) (2+\gamma)^2 \left(-8+(-1+\gamma)\,\gamma\,(4+\gamma)\right)}{1024\, b^2\, k\, (-2+\gamma^2)^2},\ -\frac{(a-c)^4 (-1+\gamma) \left(-4+(-2+\gamma)\,\gamma\right)^2 \left(-8+(-1+\gamma)\,\gamma\,(4+\gamma)\right)}{2048\, b^2\, k\, (-2+\gamma^2)^3}\right\}$$

In addition, we obtain the upstream profit and consumer, producer, and total surpluses under simultaneous pricing are as follows.

```
{πM, CS, PS, TS};
% /. Demand;
% /. OutcomepB;
% /. Outcomew;
% /. OutcomeθB // Simplify;
                    簡単な形式に
```

```
{πMB, CSB, PSB, TSB} = %
```

$$\left\{\frac{(a-c)^4}{16\, b^2\, k\, (-2+\gamma)^2},\ -\frac{(a-c)^4}{16\, b^2\, k\, (-2+\gamma)^3},\ \frac{(a-c)^4 (-4+3\gamma)}{16\, b^2\, k\, (-2+\gamma)^3},\ \frac{(a-c)^4 (-5+3\gamma)}{16\, b^2\, k\, (-2+\gamma)^3}\right\}$$

Those under sequential pricing are as follows.

```
{πM, CS, PS, TS};
% /. Demand;
% /. OutcomepF;
% /. OutcomepL;
% /. Outcomew;
% /. OutcomeθS // FullSimplify;
```
完全に簡約

```
{πMS, CSS, PSS, TSS} = %
```

$$\left\{\frac{(a-c)^4\left(-8+(-1+\gamma)\gamma(4+\gamma)\right)^2}{1024\,b^2\,k\left(-2+\gamma^2\right)^2},\right.$$
$$\frac{(a-c)^4\left(-8+(-1+\gamma)\gamma(4+\gamma)\right)\left(32+\gamma\left(32+\gamma\left(-16+\gamma\left(-20+\gamma+3\gamma^2\right)\right)\right)\right)}{4096\,b^2\,k\left(-2+\gamma^2\right)^3},$$
$$\frac{(a-c)^4\left(-8+(-1+\gamma)\gamma(4+\gamma)\right)\left(64+(-2+\gamma)\gamma(2+\gamma)\left(-4+\gamma(17+3\gamma)\right)\right)}{2048\,b^2\,k\left(-2+\gamma^2\right)^3},$$
$$\left.\frac{1}{4096\,b^2\,k\left(-2+\gamma^2\right)^3}(a-c)^4\left(-8+(-1+\gamma)\gamma(4+\gamma)\right)\left(160+\gamma\left(64+\gamma\left(-152+\gamma\left(-52+\gamma(35+9\gamma)\right)\right)\right)\right)\right\}$$

Stage 1

Proof of Proposition 1

We show the following profit ranking in (9).

$\pi^B \leq \pi^L < \pi^F$ if $0.786 \leq \gamma < 1$,
$\pi^L \leq \pi^B < \pi^F$ if $0.375 \leq \gamma < 0.786$,
$\pi^L \leq \pi^F < \pi^B$ if $0 < \gamma < 0.375$.

First, we compare π^F with π^L.

```
πF - πL // Simplify
```
簡単な形式に

$$-\frac{(a-c)^4(-1+\gamma)\gamma^3(4+3\gamma)\left(-8-4\gamma+3\gamma^2+\gamma^3\right)}{2048\,b^2\,k\left(-2+\gamma^2\right)^3}$$

Hence, we have $\pi^F > \pi^L$.

Next, we consider $\pi^B - \pi^L$.

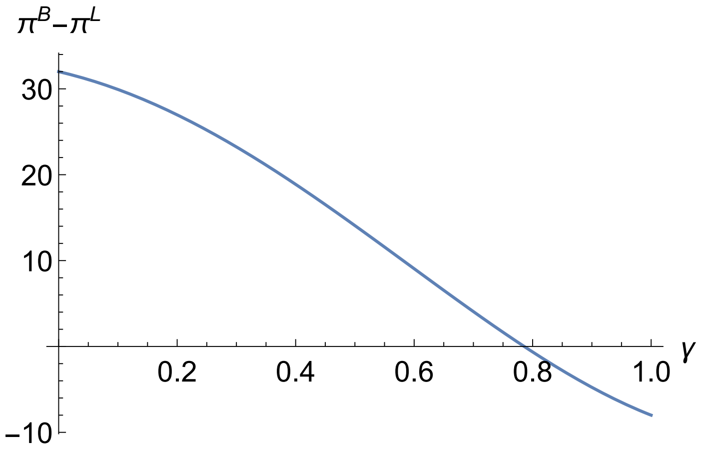
```
πB - πL // Simplify
```
簡単な形式に

$$-\frac{(a-c)^4(-1+\gamma)\gamma^2\left(-32+16\gamma+48\gamma^2-8\gamma^3-18\gamma^4+\gamma^5+\gamma^6\right)}{1024\,b^2\,k(-2+\gamma)^3\left(-2+\gamma^2\right)^2}$$

The sign of $\pi^B - \pi^L$ only depends on the term $-\left(-32+16\gamma+48\gamma^2-8\gamma^3-18\gamma^4+\gamma^5+\gamma^6\right)$.
Hence, the condition for $\pi^B - \pi^L > 0$ is as follows.

```
- (- 32 + 16 γ + 48 γ^2 - 8 γ^3 - 18 γ^4 + γ^5 + γ^6);
NSolve[% == 0, Reals]
数値解        実数領域
Plot[%%, {γ, 0, 1}, AxesLabel → {γ, "π^B-π^L"}, LabelStyle → Directive[15]]
プロット        軸のラベル        ラベルスタイル 指示子
```

{{γ → -4.24241}, {γ → 0.785753}, {γ → 1.5127}, {γ → 3.57078}}



Then, $\pi^B - \pi^L > 0$ if $\gamma < 0.786$.

Finally, we compare π^B with π^F.

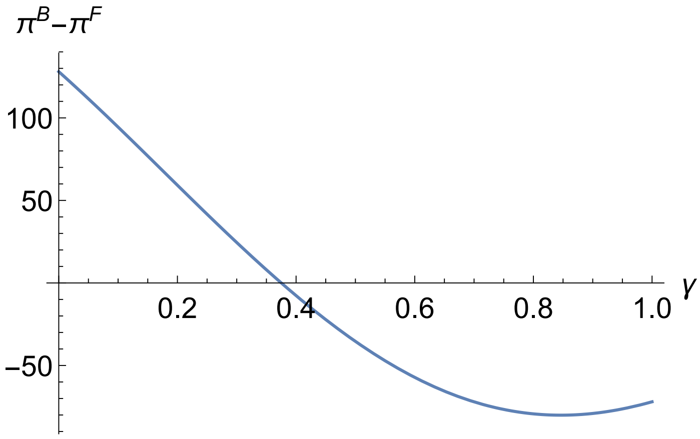
```
πB - πF // Factor // Simplify
          因数分解    簡単な形式に
```

$$\left((a-c)^4 (-1+\gamma)\gamma^2 \left(-128+320\gamma+192\gamma^2-352\gamma^3-64\gamma^4+112\gamma^5-2\gamma^6-7\gamma^7+\gamma^8\right)\right) \Big/ \left(2048\, b^2\, k\, (-2+\gamma)^3 \left(-2+\gamma^2\right)^3\right)$$

The sign of $\pi^B - \pi^F$ only depends on the term
$-\left(-128+320\gamma+192\gamma^2-352\gamma^3-64\gamma^4+112\gamma^5-2\gamma^6-7\gamma^7+\gamma^8\right)$.
Hence, the condition for $\pi^B - \pi^F > 0$ is as follows.

```
- (-128 + 320 γ + 192 γ^2 - 352 γ^3 - 64 γ^4 + 112 γ^5 - 2 γ^6 - 7 γ^7 + γ^8);
NSolve[% == 0, Reals]
数値解        実数領域
Plot[%%, {γ, 0, 1}, AxesLabel → {γ, "π^B-π^F"}, LabelStyle → Directive[15]]
プロット              軸のラベル                 ラベルスタイル 指示子
```

{{γ → -3.14907}, {γ → 0.375032}}



Then, $\pi^B - \pi^F > 0$ if $\gamma < 0.375$.

Summarizing the above results, we obtain the profit ranking in (9).

From this profit ranking, we directly obtain Proposition 1. **Q.E.D.**

Here, we derive a probability x in mixed strategy equilibrium.
Solving $x\pi^B + (1 - x)\,\pi^L = x\pi^F + (1 - x)\,\pi^B$ for x, we obtain the following probability.

```
Solve[x πB + (1 - x) πL == x πF + (1 - x) πB, x] // Flatten // Simplify
解く                                              平滑化      簡単な形式に
```

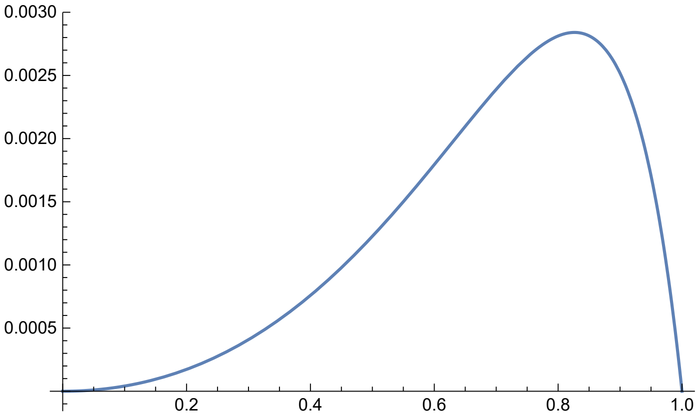
$$\left\{x \to \frac{2\left(-2+\gamma^2\right)\left(-32+16\gamma+48\gamma^2-8\gamma^3-18\gamma^4+\gamma^5+\gamma^6\right)}{256-384\gamma-448\gamma^2+416\gamma^3+232\gamma^4-132\gamma^5-38\gamma^6+9\gamma^7+\gamma^8}\right\}$$

FInally, we compare the manufacturer's profit and the various surplus measures under simultaneous and sequential pricing.

```
πMB - πMS // Factor // Simplify
          因数分解      簡単な形式に
% b^2 k / (a - c)^4;
NSolve[% == 0, Reals]
数値解         実数領域
Plot[%%, {γ, 0, 1}]
プロット
```

$$-\frac{(a-c)^4\,(-1+\gamma)\,\gamma^2\,(2+\gamma)\,\left(32-18\,\gamma^2+\gamma^3+\gamma^4\right)}{1024\,b^2\,k\,(-2+\gamma)^2\,\left(-2+\gamma^2\right)^2}$$

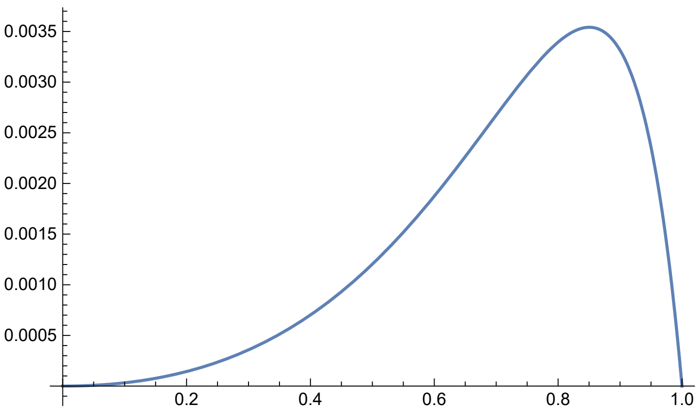
{{γ → -4.5904}, {γ → -2.}, {γ → -1.35127},
{γ → 0.}, {γ → 1.}, {γ → 1.49815}, {γ → 3.44352}}




```
CSB - CSS // Factor // Simplify
          因数分解      簡単な形式に
% b^2 k / (a - c)^4;
NSolve[% == 0, Reals]
数値解         実数領域
Plot[%%, {γ, 0, 1}]
プロット
```

$$-\frac{(a-c)^4\,(-1+\gamma)\,\gamma^2\,\left(768+384\,\gamma-896\,\gamma^2-384\,\gamma^3+360\,\gamma^4+108\,\gamma^5-58\,\gamma^6-5\,\gamma^7+3\,\gamma^8\right)}{4096\,b^2\,k\,(-2+\gamma)^3\,\left(-2+\gamma^2\right)^3}$$

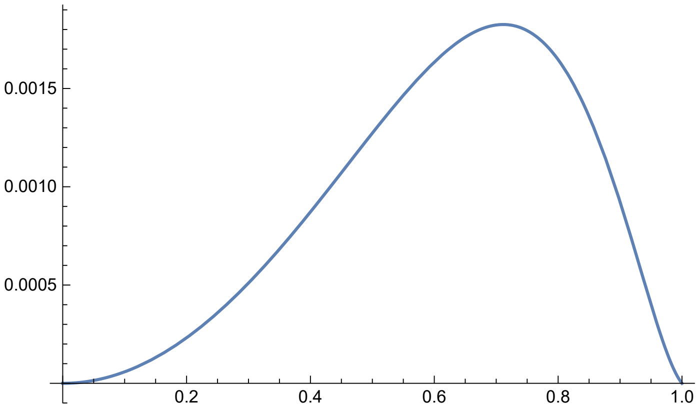
{{γ → -3.63811}, {γ → -1.31121}, {γ → 0.}, {γ → 1.}}



```
PSB - PSS // Factor // Simplify
            |因数分解   |簡単な形式に
% b^2 k / (a - c) ^4;
NSolve[% == 0, Reals]
|数値解        |実数領域
Plot[%%, {γ, 0, 1}]
|プロット
```

$$-\frac{(a-c)^4 (-1+\gamma) \gamma^2 (768 - 384\gamma - 1120\gamma^2 + 432\gamma^3 + 528\gamma^4 - 144\gamma^5 - 86\gamma^6 + 11\gamma^7 + 3\gamma^8)}{2048 b^2 k (-2+\gamma)^3 (-2+\gamma^2)^3}$$

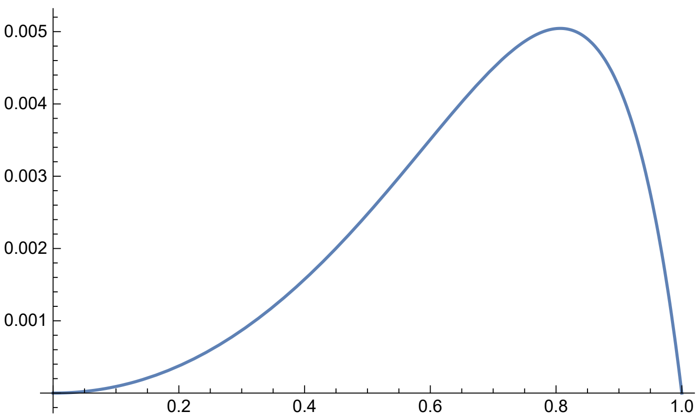
{{γ → -6.41887}, {γ → -2.41668}, {γ → 0.}, {γ → 1.}, {γ → 1.02436}, {γ → 3.79742}}




```
TSB - TSS // Factor // Simplify
            |因数分解   |簡単な形式に
% b^2 k / (a - c) ^4;
NSolve[% == 0, Reals]
|数値解        |実数領域
Plot[%%, {γ, 0, 1}]
|プロット
```

$$-\left(\left((a-c)^4 (-1+\gamma) \gamma^2 (2304 - 384\gamma - 3136\gamma^2 + 480\gamma^3 + 1416\gamma^4 - 180\gamma^5 - 230\gamma^6 + 17\gamma^7 + 9\gamma^8)\right) \Big/ \left(4096 b^2 k (-2+\gamma)^3 (-2+\gamma^2)^3\right)\right)$$

{{γ → -5.11697}, {γ → -2.18668}, {γ → 0.}, {γ → 1.},
{γ → 1.31809}, {γ → 1.54455}, {γ → 1.66468}, {γ → 3.61843}}



Therefore, we find that upstream profit and consumer, producer, and total surpluses under simultaneous pricing are higher than those under sequential pricing.

4. Extensions

4.1. Third-degree price discrimination in wholesale market

Stage 4

First, we consider the case with simultaneous pricing.
In the fourth stage, the first-order condition leads to the following outcomes.

```
{π1 /. {w → w1}, π2 /. {w → w2}};
% /. Demand;
Solve[
解く
    {D[%[[1]], p1] == 0, D[%[[2]], p2] == 0}, {p1, p2}
     微分係数              微分係数
  ] // Flatten // Simplify;
        平滑化       簡単な形式に
OutcomepBD = %
```

$$\left\{p1 \to \frac{-2\,w1 - w2\,\gamma + a\,(-2+\gamma+\gamma^2)}{-4+\gamma^2},\; p2 \to \frac{-2\,w2 - w1\,\gamma + a\,(-2+\gamma+\gamma^2)}{-4+\gamma^2}\right\}$$

Next, we consider the case with sequential pricing.
The follower chooses $p_j^{FD}(p_i^{LD})$ as follows.

```
π2 /. {w → w2};
% /. Demand;
Solve[D[%, p2] == 0, p2] // Flatten;
解く  微分係数               平滑化
OutcomepFD = %
```

```
% /. {p2 → p_j^LD, p1 → p_i^LD, w2 → wj}
```

$$\left\{p2 \to \frac{1}{2}\,(a + w2 - a\,\gamma + p1\,\gamma)\right\}$$

$$\left\{p_j^{LD} \to \frac{1}{2}\,(a + wj - a\,\gamma + \gamma\,p_i^{LD})\right\}$$

The leader sets p_i^{LD} as follows.

```
π1 /. {w → w1};
% /. Demand;
% /. OutcomepFD;
Solve[D[%, p1] == 0, p1] // Flatten // Simplify;
解く   微分係数              平滑化   簡単な形式に
OutcomepLD = %
```

```
% /. {p1 → p_i^LD, w1 → wi, w2 → wj}
```

$$\left\{p1 \to \frac{-w2\,\gamma + w1\,(-2+\gamma^2) + a\,(-2+\gamma+\gamma^2)}{2\,(-2+\gamma^2)}\right\}$$

$$\left\{p_i^{LD} \to \frac{-wj\,\gamma + wi\,(-2+\gamma^2) + a\,(-2+\gamma+\gamma^2)}{2\,(-2+\gamma^2)}\right\}$$

Stage 3

Under simultaneous pricing, the manufacturer chooses wholesale prices as follows.

```
(w1 - c) Q1 + (w2 - c) Q2 - k θ^2;
% /. Demand;
% /. OutcomepBD;
Solve[
解く
    {D[%, w1] == 0, D[%, w2] == 0}, {w1, w2}
     微分係数         微分係数
  ] // Flatten // Simplify
       平滑化      簡単な形式に
```

$$\left\{w1 \to \frac{a+c}{2},\ w2 \to \frac{a+c}{2}\right\}$$

Under sequential pricing, the manufacturer sets the following wholesale prices.

```
(w1 - c) Q1 + (w2 - c) Q2 - k θ^2;
% /. Demand;
% /. OutcomepFD;
% /. OutcomepLD;
Solve[
解く
    {D[%, w1] == 0, D[%, w2] == 0}, {w1, w2}
     微分係数         微分係数
  ] // Flatten // Simplify
       平滑化      簡単な形式に
```

$$\left\{w1 \to \frac{a+c}{2},\ w2 \to \frac{a+c}{2}\right\}$$

4.2. Timing of advertising

Case (i)

We define the following profits.

```
π1;
% /. Demand;
% /. OutcomepB // Simplify;
```
簡単な形式に

```
πBT = %
```

$$-\frac{(a-w)^2(-1+\gamma)\,\theta}{b\,(-2+\gamma)^2}$$

```
{π1, π2};
% /. Demand;
% /. OutcomepF;
% /. OutcomepL // Simplify;
```
簡単な形式に

```
{πLT, πFT} = %
```

$$\left\{\frac{(a-w)^2(-1+\gamma)\,(2+\gamma)^2\,\theta}{8\,b\,(-2+\gamma^2)},\ -\frac{(a-w)^2(-1+\gamma)\,(-4-2\gamma+\gamma^2)^2\,\theta}{16\,b\,(-2+\gamma^2)^2}\right\}$$

We consider profit ranking.

```
πFT - πLT // Simplify
```
簡単な形式に

$$-\frac{(a-w)^2(-1+\gamma)\,\gamma^3\,(4+3\gamma)\,\theta}{16\,b\,(-2+\gamma^2)^2}$$

Hence, we have $\pi^{FT} > \pi^{LT}$.

```
πLT - πBT // Simplify
```
簡単な形式に

$$\frac{(a-w)^2(-1+\gamma)\,\gamma^4\,\theta}{8\,b\,(-2+\gamma)^2\,(-2+\gamma^2)}$$

Hence, we have $\pi^{LT} > \pi^{BT}$.
Therefore, we obtain $\pi^{BT} < \pi^{LT} < \pi^{FT}$.

4.3. Persuasive advertising

We define the following demand functions.

$$\text{Demandpa} = \left\{Q1 \to \frac{(a+\theta)}{b} - \frac{p1-\gamma\,p2}{b\,(1-\gamma)},\ Q2 \to \frac{(a+\theta)}{b} - \frac{p2-\gamma\,p1}{b\,(1-\gamma)}\right\}$$

$$\left\{Q1 \to -\frac{p1-p2\,\gamma}{b\,(1-\gamma)} + \frac{a+\theta}{b},\ Q2 \to -\frac{p2-p1\,\gamma}{b\,(1-\gamma)} + \frac{a+\theta}{b}\right\}$$

We define z^{SOC} as follows.

$$\text{zSOC} = 1 / (2\,(2-\gamma))$$

$$\frac{1}{2\,(2-\gamma)}$$

Stage 4

Under simultaneous pricing, the first-order conditions yields p^{Bpa} as follows.

```
{π1, π2};
% /. Demandpa;
Solve[
解く
    {D[%[[1]], p1] == 0, D[%[[2]], p2] == 0}, {p1, p2}
     微分係数            微分係数
  ] // Flatten // Simplify;
       平滑化       簡単な形式に
OutcomepBpa = %

% /. {p1 → p^Bpa, p2 → p^Bpa}
```

$$\left\{p1 \to \frac{a + w - a\gamma + \theta - \gamma\theta}{2-\gamma},\ p2 \to \frac{a + w - a\gamma + \theta - \gamma\theta}{2-\gamma}\right\}$$

$$\left\{p^{\mathrm{Bpa}} \to \frac{a + w - a\gamma + \theta - \gamma\theta}{2-\gamma},\ p^{\mathrm{Bpa}} \to \frac{a + w - a\gamma + \theta - \gamma\theta}{2-\gamma}\right\}$$

Under sequential pricing, the follower chooses $p^{\mathrm{Fpa}}(p^{\mathrm{Lpa}})$ as follows.

```
π2;
% /. Demandpa;
Solve[D[%, p2] == 0, p2] // Flatten // Simplify;
解く  微分係数                平滑化       簡単な形式に
OutcomepFpa = %

% /. {p1 → p^Lpa, p2 → p^Fpa}
```

$$\left\{p2 \to \frac{1}{2}\left(a + w - a\gamma + p1\,\gamma + \theta - \gamma\theta\right)\right\}$$

$$\left\{p^{\mathrm{Fpa}} \to \frac{1}{2}\left(a + w - a\gamma + p^{\mathrm{Lpa}}\gamma + \theta - \gamma\theta\right)\right\}$$

The leader sets p^{Lpa} as follows.

```
π1;
% /. Demandpa;
% /. OutcomepFpa;
Solve[D[%, p1] == 0, p1] // Flatten // Simplify;
解く  微分係数                平滑化       簡単な形式に
OutcomepLpa = %

% /. {p1 → p^Lpa}
```

$$\left\{p1 \to \frac{w\left(-2-\gamma+\gamma^2\right) + a\left(-2+\gamma+\gamma^2\right) + \left(-2+\gamma+\gamma^2\right)\theta}{2\left(-2+\gamma^2\right)}\right\}$$

$$\left\{p^{\mathrm{Lpa}} \to \frac{w\left(-2-\gamma+\gamma^2\right) + a\left(-2+\gamma+\gamma^2\right) + \left(-2+\gamma+\gamma^2\right)\theta}{2\left(-2+\gamma^2\right)}\right\}$$

Stage 3

Under simultaneous pricing, the manufacturer chooses the following wholesale price.

```
πM;
% /. Demandpa;
% /. OutcomepBpa;
Solve[D[%, w] == 0, w] // Flatten // Simplify;
解く   微分係数            平滑化     簡単な形式に
Outcomewpa = %
```

$\left\{w \to \frac{1}{2}(a + c + \theta)\right\}$

Under sequential pricing, the manufacturer sets the following wholesale price.

```
πM;
% /. Demandpa;
% /. OutcomepFpa;
% /. OutcomepLpa;
Solve[D[%, w] == 0, w] // Flatten // Simplify
解く   微分係数            平滑化     簡単な形式に
```

$\left\{w \to \frac{1}{2}(a + c + \theta)\right\}$

Stage 2

Under simultaneous pricing, the manufacturer chooses the advertising level θ^{Bpa} as follows.

```
πM;
% /. Demandpa;
% /. OutcomepBpa;
% /. Outcomewpa;
Solve[D[%, θ] == 0, θ] // Flatten // Simplify;
解く   微分係数            平滑化     簡単な形式に
% /. {b → z / k};
OutcomeθBpa = %
θBpa = θ /. %;

%% /. {θ → θ^Bpa}
```

$\left\{\theta \to \frac{-a + c}{1 + 2z(-2 + \gamma)}\right\}$

$\left\{\theta^{Bpa} \to \frac{-a + c}{1 + 2z(-2 + \gamma)}\right\}$

Under sequential pricing, the manufacturer sets the advertising level θ^{Spa} as follows.

```
πM;
% /. Demandpa;
% /. OutcomepFpa;
% /. OutcomepLpa;
% /. Outcomewpa;
Solve[D[%, θ] == 0, θ] // Flatten // FullSimplify;
解く  微分係数  平滑化  完全に簡約
% /. {b → z / k};
OutcomeθSpa = %
θSpa = θ /. %;

%% /. {θ → θ^Spa}
```

$$\left\{\theta \to -\frac{(a-c)\left(-8+(-1+\gamma)\,\gamma\,(4+\gamma)\right)}{-8+(-1+\gamma)\,\gamma\,(4+\gamma)-16\,z\,(-2+\gamma^2)}\right\}$$

$$\left\{\theta^{Spa} \to -\frac{(a-c)\left(-8+(-1+\gamma)\,\gamma\,(4+\gamma)\right)}{-8+(-1+\gamma)\,\gamma\,(4+\gamma)-16\,z\,(-2+\gamma^2)}\right\}$$

Proof of Lemma 3

We compare θ^{Bpa} with θ^{Spa}.

```
θBpa - θSpa // FullSimplify
完全に簡約

Reduce[% > 0 && a > c > 0 && 0 < γ < 1 && z > zSOC] // Factor
簡約  因数分解
```

$$-\frac{2\,(a-c)\,z\,\gamma^2\,(-2+\gamma+\gamma^2)}{(1+2\,z\,(-2+\gamma))\,\left(8-(-1+\gamma)\,\gamma\,(4+\gamma)+16\,z\,(-2+\gamma^2)\right)}$$

$$0<\gamma<1\ \&\&\ z>-\frac{1}{2\,(-2+\gamma)}\ \&\&\ c>0\ \&\&\ a>c$$

Therefore, we obtain $\theta^{Bpa} > \theta^{Spa}$. **Q.E.D.**

Substituting the subgame outcomes into the retailers' profits, the profit of retailer with simultaneous pricing is π^{Bpa}:

```
π1;
% /. Demandpa;
% /. OutcomepBpa;
% /. Outcomewpa;
% /. OutcomeθBpa // Simplify;
簡単な形式に
% /. {z^2 → b ξ / ((a - c)^2 (1 - γ))} // Simplify;
簡単な形式に

πBpa = %
```

$$\frac{\xi}{(1+2\,z\,(-2+\gamma))^2}$$

The profits of leader and follower with sequential pricing are π^{Lpa} and π^{Fpa}, respectively:

```
{π1, π2};
% /. Demandpa;
% /. OutcomepFpa;
% /. OutcomepLpa;
% /. Outcomewpa;
% /. OutcomeθSpa // Simplify;
                   簡単な形式に
% /. {z^2 → b ξ / ((a - c) ^2 (1 - γ))} // FullSimplify;
                                         完全に簡約
{πLpa, πFpa} = %
```

$$\left\{-\frac{8\left(2+\gamma\right)^{2}\left(-2+\gamma^{2}\right)\xi}{\left(8-\left(-1+\gamma\right)\gamma\left(4+\gamma\right)+16\,z\left(-2+\gamma^{2}\right)\right)^{2}},\ \frac{4\left(-4+\left(-2+\gamma\right)\gamma\right)^{2}\xi}{\left(8-\left(-1+\gamma\right)\gamma\left(4+\gamma\right)+16\,z\left(-2+\gamma^{2}\right)\right)^{2}}\right\}$$

Proof of Proposition 2

Comparing π^{Bpa}, π^{Lpa}, and π^{Fpa}, we show the following profit ranking.

$\pi^{\text{Lpa}} < \pi^{\text{Fpa}} < \pi^{\text{Bpa}}$ if $z^{\text{SOC}} < z < \overline{z}^{\text{BF}}$,
$\pi^{\text{Lpa}} < \pi^{\text{Bpa}} < \pi^{\text{Fpa}}$ if $\overline{z}^{\text{BF}} < z < \overline{z}^{\text{BL}}$,
$\pi^{\text{Bpa}} < \pi^{\text{Lpa}} < \pi^{\text{Fpa}}$ if $\overline{z}^{\text{BF}} < z < \overline{z}^{\text{BL}}$.

First, we consider the sign of $\pi^{\text{Fpa}} - \pi^{\text{Lpa}}$.

```
πFpa - πLpa // FullSimplify
               完全に簡約
```

$$\frac{4\gamma^{3}\left(4+3\gamma\right)\xi}{\left(8-\left(-1+\gamma\right)\gamma\left(4+\gamma\right)+16\,z\left(-2+\gamma^{2}\right)\right)^{2}}$$

Then, we obtain $\pi^{\text{Fpa}} > \pi^{\text{Lpa}}$.

Next, we consider the sign of $\pi^{\text{Bpa}} - \pi^{\text{Lpa}}$.

```
πBpa - πLpa // Factor // Simplify
               因数分解    簡単な形式に
```

$$\frac{\gamma^{2}\left(-16-8\gamma+9\gamma^{2}+6\gamma^{3}+\gamma^{4}-32\,z\left(-2+\gamma^{2}\right)+32\,z^{2}\gamma^{2}\left(-2+\gamma^{2}\right)\right)\xi}{\left(1+2\,z\left(-2+\gamma\right)\right)^{2}\left(-8-4\gamma+3\gamma^{2}+\gamma^{3}-16\,z\left(-2+\gamma^{2}\right)\right)^{2}}$$

The sign of $\pi^{\text{Bpa}} - \pi^{\text{Lpa}}$ only depends on the terms in the numerator:
$-16-8\gamma+9\gamma^{2}+6\gamma^{3}+\gamma^{4}-32\,z\left(-2+\gamma^{2}\right)+32\,z^{2}\gamma^{2}\left(-2+\gamma^{2}\right)$.
Solving $-16-8\gamma+9\gamma^{2}+6\gamma^{3}+\gamma^{4}-32\,z\left(-2+\gamma^{2}\right)+32\,z^{2}\gamma^{2}\left(-2+\gamma^{2}\right)=0$ for z, we obtain the threshold values $\underline{z}^{\text{BL}}$ (= zBLl) and $\overline{z}^{\text{BL}}$ (= zBLh) as follows.

```
Solve[-16 - 8 γ + 9 γ^2 + 6 γ^3 + γ^4 - 32 z (-2 + γ^2) + 32 z^2 γ^2 (-2 + γ^2) == 0, z];
解く
Simplify[%, 0 < γ < 1]
簡単な形式に

z /. %;

{zBLh, zBLl} = %
```

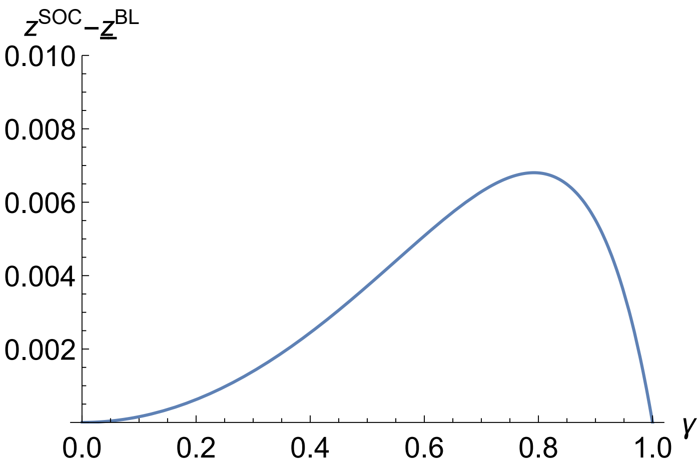
$$\left\{\left\{z \to \frac{-8+4\gamma^2+(-1+\gamma)(2+\gamma)^2\sqrt{4-2\gamma^2}}{8\gamma^2(-2+\gamma^2)}\right\}, \left\{z \to -\frac{8-4\gamma^2+(-1+\gamma)(2+\gamma)^2\sqrt{4-2\gamma^2}}{8\gamma^2(-2+\gamma^2)}\right\}\right\}$$

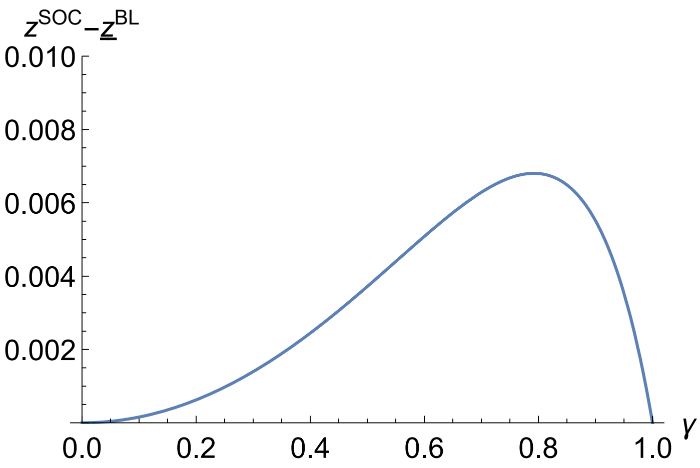
$$\left\{\frac{-8+4\gamma^2+(-1+\gamma)(2+\gamma)^2\sqrt{4-2\gamma^2}}{8\gamma^2(-2+\gamma^2)}, -\frac{8-4\gamma^2+(-1+\gamma)(2+\gamma)^2\sqrt{4-2\gamma^2}}{8\gamma^2(-2+\gamma^2)}\right\}$$

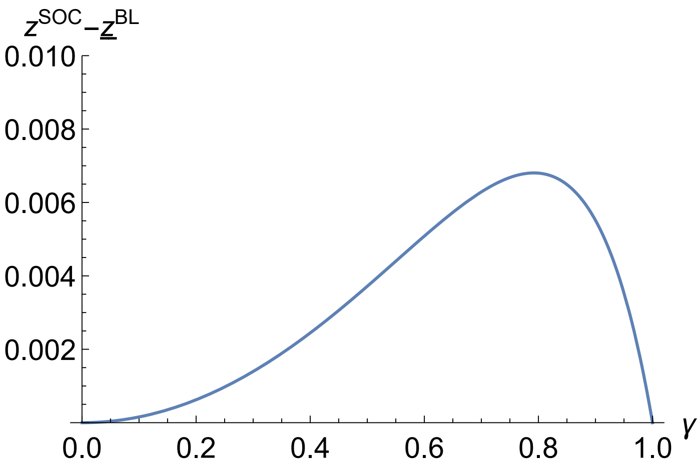
Comparing $\underline{z}^{BL}$ with z^{SOC}, we have $\underline{z}^{BL} < z^{SOC}$.

```
zSOC - zBLl;
NSolve[% == 0, Reals]
数値解      実数領域
Plot[%%, {γ, 0, 1}, PlotRange → {0, 0.01},
プロット              プロット範囲
 AxesLabel → {γ, "z^SOC-z^BL"}, LabelStyle → Directive[15]]
 軸のラベル                     ラベルスタイル 指示子
```

$\{\{\gamma \to 1.\}\}$





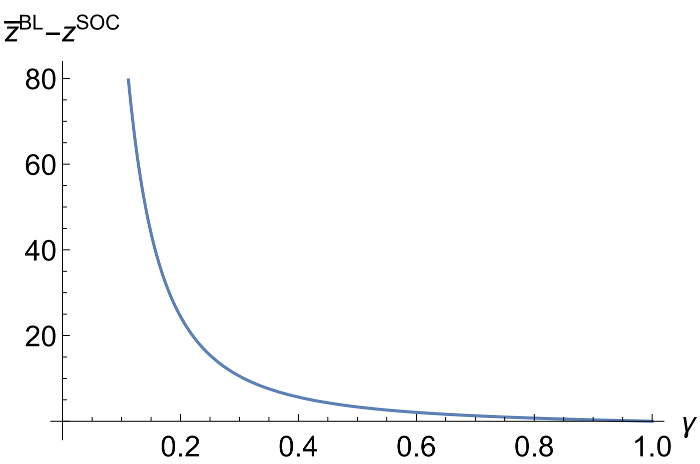


Next, we show $z^{SOC} < \overline{z}^{BL}$.

```
zBLh - zSOC;

NSolve[% == 0, Reals]
数値解           実数領域
Plot[%%, {γ, 0, 1}, AxesLabel → {γ, "z̄^BL-z^SOC"}, LabelStyle → Directive[15]]
プロット                軸のラベル                  ラベルスタイル  指示子
```

{{γ → 1.}}



Hence, we obtain $\underline{z}^{BL} < z^{SOC} < \overline{z}^{BL}$.

Summarizing the above results, we have

$\pi^{Bpa} > \pi^{Lpa}$ if $z^{SOC} < z < \overline{z}^{BL}$,
$\pi^{Bpa} \leq \pi^{Lpa}$ if $z \geq \overline{z}^{BL}$.

FInally, we compare π^{Bpa} with π^{Fpa}.

```
πBpa - πFpa // Factor // FullSimplify
               因数分解    完全に簡約
```

$$-\frac{\gamma^2 \left(-1+(-1+4z)\gamma\right)\left(-16+4z\left(16+(-8+\gamma)\gamma^2\right)+\gamma\left(-8+\gamma(5+\gamma)\right)\right)\xi}{\left(1+2z(-2+\gamma)\right)^2\left(8-(-1+\gamma)\gamma(4+\gamma)+16z\left(-2+\gamma^2\right)\right)^2}$$

The sign of $\pi^{Bpa} - \pi^{Fpa}$ only depends on the terms
$-(-1+(-1+4z)\gamma)(-16+4z(16+(-8+\gamma)\gamma^2)+\gamma(-8+\gamma(5+\gamma)))$.
Solving $-(-1+(-1+4z)\gamma)(-16+4z(16+(-8+\gamma)\gamma^2)+\gamma(-8+\gamma(5+\gamma)))=0$ for z, we obtain two threshold values: $\underline{z}^{BF}$ (= zBFl) and $\overline{z}^{BF}$ (= zBFh).

```
Solve[-(-1 + (-1 + 4 z) γ) (-16 + 4 z (16 + (-8 + γ) γ^2) + γ (-8 + γ (5 + γ))) == 0, z]
解く

z /. %;
{zBFh, zBFl} = %
```

$$\left\{\left\{z\to\frac{1+\gamma}{4\gamma}\right\},\left\{z\to\frac{16+8\gamma-5\gamma^2-\gamma^3}{4\left(16-8\gamma^2+\gamma^3\right)}\right\}\right\}$$

$$\left\{\frac{1+\gamma}{4\gamma},\frac{16+8\gamma-5\gamma^2-\gamma^3}{4\left(16-8\gamma^2+\gamma^3\right)}\right\}$$

Here, we show $\underline{z}^{BF} < z^{SOC} < \overline{z}^{BF}$.

```
zSOC - zBFl // Simplify
               簡単な形式に
```

$$\frac{\gamma^2\left(-2+\gamma+\gamma^2\right)}{4(-2+\gamma)\left(16-8\gamma^2+\gamma^3\right)}$$

```
zBFh - zSOC // Simplify
               簡単な形式に
```

$$\frac{-2+\gamma+\gamma^2}{4(-2+\gamma)\gamma}$$

Hence, we omit the threshold value $\underline{z}^{BF}$ (= zBFl).
Summarizing the above discussion, we obtain the followings.

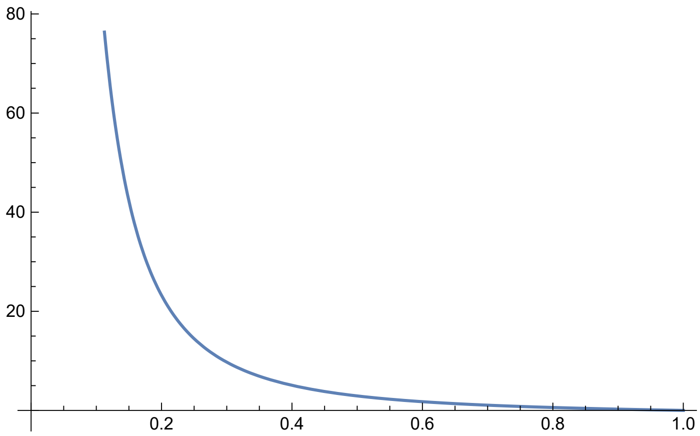
$\pi^{Bpa} > \pi^{Fpa}$ if $z^{SOC} < z < \overline{z}^{BF}$,
$\pi^{Bpa} \leq \pi^{Fpa}$ if $z \geq \overline{z}^{BF}$.

Before obtaining profit ranking, we confirm $\overline{z}^{BF} < \overline{z}^{BL}$.

```
zBLh - zBFh // Factor // FullSimplify
              因数分解      完全に簡約
NSolve[% == 0]
数値解
Plot[%%, {γ, 0, 1}]
プロット
```

$$\frac{\left(-2+\gamma+\gamma^2\right)\left(4+2\sqrt{4-2\gamma^2}+\gamma\left(-2\gamma+\sqrt{4-2\gamma^2}\right)\right)}{8\gamma^2\left(-2+\gamma^2\right)}$$

{{γ → -2.}, {γ → 1.}}



Summarizing all the cases, we obtain the following profit ranking.

$\pi^{\text{Lpa}} < \pi^{\text{Fpa}} < \pi^{\text{Bpa}}$ if $z^{\text{SOC}} < z < \overline{z}^{\text{BF}}$,
$\pi^{\text{Lpa}} < \pi^{\text{Bpa}} < \pi^{\text{Fpa}}$ if $\overline{z}^{\text{BF}} < z < \overline{z}^{\text{BL}}$,
$\pi^{\text{Bpa}} < \pi^{\text{Lpa}} < \pi^{\text{Fpa}}$ if $\overline{z}^{\text{BF}} < z < \overline{z}^{\text{BL}}$.

Q.E.D.

4.4. Cournot competition between retailers

We define the inverse demand functions.

$$\text{IDemand} = \left\{p1 \to a - \frac{b\,(Q1+\gamma\,Q2)}{\theta\,(1+\gamma)},\ p2 \to a - \frac{b\,(Q2+\gamma\,Q1)}{\theta\,(1+\gamma)}\right\}$$

$$\left\{p1 \to a - \frac{b\,(Q1+Q2\,\gamma)}{(1+\gamma)\,\theta},\ p2 \to a - \frac{b\,(Q2+Q1\,\gamma)}{(1+\gamma)\,\theta}\right\}$$

Stage 4

Under simultaneous competition, the first-order condition leads to the output Q^C as follows.

```
{π1, π2};
% /. IDemand;
Solve[
解く
    {D[%[[1]], Q1] == 0, D[%[[2]], Q2] == 0}, {Q1, Q2}
     微分係数           微分係数
  ] // Flatten // Simplify;
       平滑化      簡単な形式に
OutcomeQC = %
```

$$\left\{Q1 \to \frac{(a-w)\,(1+\gamma)\,\theta}{b\,(2+\gamma)},\ Q2 \to \frac{(a-w)\,(1+\gamma)\,\theta}{b\,(2+\gamma)}\right\}$$

Next, under sequential competition, the follower chooses $Q^{FC}(Q^{LC})$.

```
π2;
% /. IDemand;
Solve[D[%, Q2] == 0, Q2] // Flatten // Simplify;
解く  微分係数                平滑化      簡単な形式に
OutcomeQFC = %

% /. {Q1 → Q^LC, Q2 → Q^FC}
```

$$\left\{Q2 \to \frac{-b\,Q1\,\gamma + (a-w)\,(1+\gamma)\,\theta}{2\,b}\right\}$$

$$\left\{Q^{FC} \to \frac{-b\,Q^{LC}\,\gamma + (a-w)\,(1+\gamma)\,\theta}{2\,b}\right\}$$

The leader sets the following output Q^{LC}.

```
π1;
% /. IDemand;
% /. OutcomeQFC;
Solve[D[%, Q1] == 0, Q1] // Flatten // Simplify;
解く  微分係数                平滑化      簡単な形式に
OutcomeQLC = %

% /. {Q1 → Q^LC}
```

$$\left\{Q1 \to \frac{(a-w)\,(-2+\gamma)\,(1+\gamma)\,\theta}{2\,b\,(-2+\gamma^2)}\right\}$$

$$\left\{Q^{LC} \to \frac{(a-w)\,(-2+\gamma)\,(1+\gamma)\,\theta}{2\,b\,(-2+\gamma^2)}\right\}$$

Stage 3

With simultaneous competition, the manufacturer chooses the following wholesale price.

```
πM;
% /. IDemand;
% /. OutcomeQC;
Solve[D[%, w] == 0, w] // Flatten // Simplify;
解く 微分係数 平滑化 簡単な形式に
OutcomewC = %
```

$\left\{w \to \frac{a+c}{2}\right\}$

Similarly, under the sequential competition, the manufacturer sets the wholesale price as follows.

```
πM;
% /. IDemand;
% /. OutcomeQFC;
% /. OutcomeQLC;
Solve[D[%, w] == 0, w] // Flatten // Simplify
解く 微分係数 平滑化 簡単な形式に
```

$\left\{w \to \frac{a+c}{2}\right\}$

Stage 2

Under simultaneous competition, the manufacturer chooses the level of advertising θ^C.

```
πM;
% /. IDemand;
% /. OutcomeQC;
% /. OutcomewC;
Solve[D[%, θ] == 0, θ] // Flatten // Simplify;
解く 微分係数 平滑化 簡単な形式に
OutcomeθC = %

% /. {θ → θ^C}

θC = θ /. %%
```

$\left\{\theta \to \frac{(a-c)^2 (1+\gamma)}{4 b k (2+\gamma)}\right\}$

$\left\{\theta^C \to \frac{(a-c)^2 (1+\gamma)}{4 b k (2+\gamma)}\right\}$

$\frac{(a-c)^2 (1+\gamma)}{4 b k (2+\gamma)}$

Under sequential competition, the manufacturer sets θ^{SC} as follows.

```
πM;
% /. IDemand;
% /. OutcomeQFC;
% /. OutcomeQLC;
% /. OutcomewC;
Solve[D[%, θ] == 0, θ] // Flatten // FullSimplify;
解く 微分係数 平滑化 完全に簡約
OutcomeθSC = %

% /. {θ → θ^SC}

θSC = θ /. %%
```

$$\left\{\theta \to \frac{(a-c)^2 (1+\gamma) (-8+\gamma (4+\gamma))}{32 \, b \, k \, (-2+\gamma^2)}\right\}$$

$$\left\{\theta^{SC} \to \frac{(a-c)^2 (1+\gamma) (-8+\gamma (4+\gamma))}{32 \, b \, k \, (-2+\gamma^2)}\right\}$$

$$\frac{(a-c)^2 (1+\gamma) (-8+\gamma (4+\gamma))}{32 \, b \, k \, (-2+\gamma^2)}$$

Proof of Lemma 4

We compare θ^{SC} with θ^{C}.

```
θSC - θC // Simplify
簡単な形式に
```

$$\frac{(a-c)^2 (-2+\gamma) \gamma^2 (1+\gamma)}{32 \, b \, k \, (2+\gamma) (-2+\gamma^2)}$$

Hence, we obtain $\theta^{SC} > \theta^{C}$. **Q.E.D.**

Stage 1

The profit of retailer with simultaneous competition is π^{C}:

```
π1;
% /. IDemand;
% /. OutcomeQC;
% /. OutcomewC;
% /. OutcomeθC // Simplify;
簡単な形式に
πC = %
```

$$\frac{(a-c)^4 (1+\gamma)^2}{16 \, b^2 \, k \, (2+\gamma)^3}$$

The profits of leader and follower with sequential competition are π^{LC} and π^{FC}, respectively.

```
{π1, π2};
% /. IDemand;
% /. OutcomeQFC;
% /. OutcomeQLC;
% /. OutcomewC;
% /. OutcomeθSC // FullSimplify;
                  完全に簡約
{πLC, πFC} = %
```

$$\left\{-\frac{(a-c)^4 (-2+\gamma)^2 (1+\gamma)^2 (-8+\gamma (4+\gamma))}{1024 b^2 k (-2+\gamma^2)^2}, \frac{(a-c)^4 (1+\gamma)^2 (-4+\gamma (2+\gamma))^2 (-8+\gamma (4+\gamma))}{2048 b^2 k (-2+\gamma^2)^3}\right\}$$

Proof of Proposition 3

In order to obtain Proposition 3, we show the following profit ranking.

$\pi^C < \pi^{FC} < \pi^{LC}$ if $\gamma < 0.396$,
$\pi^{FC} \leq \pi^C < \pi^{LC}$ if $\gamma \geq 0.396$.

First, we compare π^{LC} with π^C.

```
πLC - πC // Simplify
            簡単な形式に
```

$$-\frac{(a-c)^4 \gamma^2 (1+\gamma)^2 (-32+16\gamma-8\gamma^3+6\gamma^4+\gamma^5)}{1024 b^2 k (2+\gamma)^3 (-2+\gamma^2)^2}$$

Hence, we have $\pi^{LC} > \pi^C$.

Next, we consider $\pi^C - \pi^{FC}$.

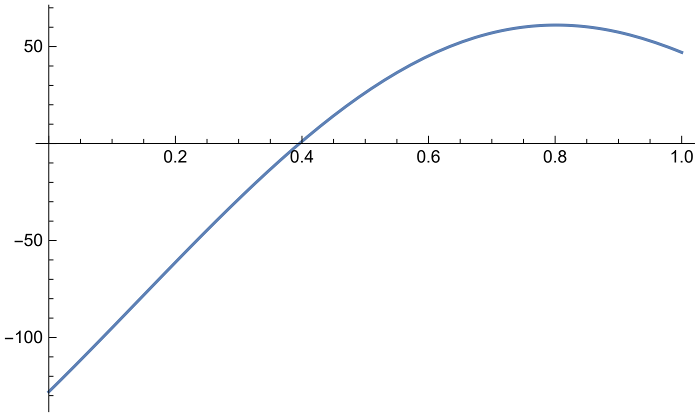
```
πC - πFC // Factor // Simplify
            因数分解    簡単な形式に
```

$$-\frac{(a-c)^4 \gamma^2 (1+\gamma)^2 (-128+320\gamma+128\gamma^2-288\gamma^3-64\gamma^4+64\gamma^5+14\gamma^6+\gamma^7)}{2048 b^2 k (2+\gamma)^3 (-2+\gamma^2)^3}$$

The sign of $\pi^C - \pi^{FC}$ only depends on the terms
$-128+320\gamma+128\gamma^2-288\gamma^3-64\gamma^4+64\gamma^5+14\gamma^6+\gamma^7$.
Solving $-128+320\gamma+128\gamma^2-288\gamma^3-64\gamma^4+64\gamma^5+14\gamma^6+\gamma^7 < 0$ for γ, we have

```
-128 + 320 γ + 128 γ^2 - 288 γ^3 - 64 γ^4 + 64 γ^5 + 14 γ^6 + γ^7;
NSolve[% == 0, Reals]
数値解          実数領域
Plot[%%, {γ, 0, 1}]
プロット
```

{{γ → 0.395954}, {γ → 1.30249}, {γ → 1.46532}}



Hence, we obtain $\pi^C - \pi^{FC} < 0$ if $\gamma < 0.396$; $\pi^C - \pi^{FC} \geq 0$ otherwise.

Finally, we compare π^{FC} with π^{LC}.

```
πLC - πFC // FullSimplify
             完全に簡約
```

$$-\frac{(a-c)^4 \gamma^3 (1+\gamma)^2 (-4+3\gamma) (-8+\gamma (4+\gamma))}{2048 b^2 k (-2+\gamma^2)^3}$$

Hence, we obtain $\pi^{LC} > \pi^{FC}$.

Summarizing the above results, we obtain the following profit ranking.

$\pi^C < \pi^{FC} < \pi^{LC}$ if $\gamma < 0.396$,
$\pi^{FC} \leq \pi^C < \pi^{LC}$ if $\gamma \geq 0.396$.

Therefore, we obtain Proposition 3. **Q.E.D.**